

ELLIPTIC SELBERG INTEGRALS

G. FELDER ^{*}, L. STEVENS [◇], AND A. VARCHENKO [◇]

^{*} *Departement Mathematik, ETH-Zentrum, 8092 Zürich, Switzerland,
felder@math.ethz.ch*

[◇] *Department of Mathematics, University of North Carolina at Chapel Hill,
Chapel Hill, NC 27599-3250, USA,
stevens@math.unc.edu, av@math.unc.edu*

March, 2001

The Selberg integral is

$$B_p(\alpha, \beta, \gamma) = \int_{\Delta_p} \prod_{j=1}^p t_j^{\alpha-1} (1-t_j)^{\beta-1} \prod_{0 \leq j < k \leq 1} (t_j - t_k)^{2\gamma},$$

where $\Delta_p = \{t \in \mathbb{R}^p \mid 0 \leq t_p \leq \dots \leq t_1 \leq 1\}$. The Selberg integral is a generalization of the beta function. It can be calculated explicitly,

$$B_p(\alpha, \beta, \gamma) = \frac{1}{p!} \prod_{j=0}^{p-1} \frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+(p+j-1)\gamma)}.$$

The Selberg integral has many applications, see [A1, A2, As, D, DF1, DF2, M, S]. In this paper, we present an elliptic version of the Selberg integral.

Let $\vartheta_1(t, \tau)$ be the first Jacobi theta function [WW],

$$\vartheta_1(t, \tau) = - \sum_{j \in \mathbb{Z}} e^{\pi i(j+\frac{1}{2})^2 \tau + 2\pi i(j+\frac{1}{2})(t+\frac{1}{2})}.$$

Introduce special functions

$$\sigma_\lambda(t, \tau) = \frac{\vartheta_1(\lambda - t, \tau) \vartheta_1'(0, \tau)}{\vartheta_1(\lambda, \tau) \vartheta_1(t, \tau)}, \quad \rho(t, \tau) = \frac{\vartheta_1'(t, \tau)}{\vartheta_1(t, \tau)}, \quad E(t, \tau) = \frac{\vartheta_1(t, \tau)}{\vartheta_1'(0, \tau)}.$$

Here ' denotes the derivative with respect to the first argument.

Let $\kappa \geq 2$ be an integer. The theta functions

$$\theta_{\kappa, m}(\lambda, \tau) = \sum_{j \in \mathbb{Z}} e^{2\pi i \kappa (j + \frac{m}{2\kappa})^2 \tau + 2\pi i \kappa (j + \frac{m}{2\kappa}) \lambda}, \quad m \in \mathbb{Z}/2\kappa\mathbb{Z},$$

form a basis of the theta functions of level κ .

For a positive integer p , the elliptic Selberg integral $I_p(\lambda, \tau)$ is the integral,

$$I_p(\lambda, \tau) = J_p(\lambda, \tau) + (-1)^{p+1} J_p(-\lambda, \tau),$$

where

$$J_p(\lambda, \tau) = \int_{\Delta_p} \prod_{j=1}^p E(t_j, \tau)^{-\frac{p}{p+1}} \prod_{1 \leq j < k \leq p} E(t_j - t_k, \tau)^{\frac{1}{p+1}} \times \\ \prod_{j=1}^p \sigma_\lambda(t_j, \tau) \theta_{2(p+1), p+1} \left(\lambda + \frac{1}{p+1} \sum_{j=1}^p t_j, \tau \right) dt_1 \dots dt_p .$$

The branch of the logarithm is chosen in such a way that $\arg(E(t, \tau)) \rightarrow 0$ as $t \rightarrow 0^+$, and the integral is understood as a natural analytic continuation.¹

Theorem 1. *We have*

$$(1) \quad I_p(\lambda, \tau) = c_p B_p \left(\frac{1}{2} + \frac{1}{2(p+1)}, -\frac{p}{p+1}, \frac{1}{2(p+1)} \right) \vartheta_1(\lambda, \tau)^{p+1}$$

where

$$c_p = -(2\pi)^{\frac{p}{2}} e^{\pi i \frac{p}{p+1}} e^{-\pi i \frac{p+2}{4}} \prod_{j=1}^p \left(1 - e^{-\pi i \frac{j}{p+1}} \right).$$

The theorem is a generalization of theorem 13 in [FV1]. The proof is based on the following remarks. Consider the heat equation

$$4\pi i(p+1) \frac{\partial u}{\partial \tau}(\lambda, \tau) = \frac{\partial^2 u}{\partial \lambda^2}(\lambda, \tau) + p(p+1) \rho'(\lambda, \tau) u(\lambda, \tau).$$

It is known that this equation has a one dimensional space of solutions $u(\lambda, \tau)$ which are holomorphic theta functions of level $2(p+1)$,

$$u(\lambda + 2, \tau) = u(\lambda, \tau), \quad u(\lambda + 2\tau, \tau) = e^{-4\pi i(p+1)(\lambda+\tau)} u(\lambda, \tau)$$

and Weyl skew-symmetric, $u(-\lambda, \tau) = (-1)^{p+1} u(\lambda, \tau)$, see [FV1, FV2]. The space is called the space of conformal blocks. Clearly the right hand side of (1) has these properties. According to [FV1], the left hand side of (1) also has these properties. Thus the two functions are proportional. The coefficient of proportionality is easily calculated in the limit $\tau \rightarrow i\infty$.

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¹To define the analytic continuation, we replace the exponential $-\frac{p}{p+1}$ by a , and consider the analytic continuation with respect to a from the region where a is positive.

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